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One-Dimensional Random Walk with Self-Interaction

H. Żołądek[1]

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A standard random walk on a one-dimensional integer lattice is considered where the probability of k self-intersections of a path $\omega = (0, \omega(1), ..., \omega(n))$ is proportional to $e^{-\lambda k}$. It is proven that for $\lambda < 0$, $n^{-1/3}\omega(n)$ converges to a certain continuous random variable. For $\lambda > 0$ the formulas are given for the asymptotic Westerwater velocity of a generic path and for the variance of the fluctuations about the asymptotic motion.

KEY WORDS: Random walk; random path; Wiener process; Edwards model.

1. INTRODUCTION

The self-avoiding random walks in $\mathbb{Z}^d$ play an important role in many physical problems: e.g., in percolation theory,[1] the theory of macromolecular solutions,[2,3] and constructive quantum field theory.[4] Also considered besides self-avoiding random walks have been random walks with a limited probability of self-intersection (or with repelling interaction).[2,3] The important mathematical problem is to describe the asymptotic behavior of a generic random path in these systems.

The following results (in a strong mathematical sense) have been obtained in this field:

1. Brydges and Spencer[5] proved that for $d \geqslant 5$ and weak interaction ($|\lambda| \ll 1$), $n^{-1/2}\omega(n)$ converges to the Gaussian random variable.

[1] Institute of Mathematics, Warsaw University, 00-901 Warsaw, Poland.

2. Westerwater[6,8] considered Edwards' model of polymer chains[3] and proved the existence of the measure

$$\text{const} \times \exp\left[-\lambda \iint \delta(x(s)-x(t))\, ds\, dt \right] dw$$

in the space of random paths of the standard three-dimensional Wiener process dw. Kusoka[10] also investigated this measure. Westerwater[7] analyzed certain approximations of the polymer model, and proved the drift phenomenon for the approximate process [see formula (1) for $d=3$ below].

3. In one dimension for $\lambda>0$ Westerwater[9] proved that the probability law of $x(T\bullet)/T$ converges to a distribution focused in two paths $t \to \pm rt$, where r is the asymptotic velocity. Kusoka[11] investigated the asymptotic behavior of the one-dimensional measure. The existence of a phase transition in one dimension was suggested by Thouless.[13]

In the low-dimension case ($d<4$) the following picture of phase transition is expected. The transition to a non-Gaussian behavior appears immediately after including the repulsive or contractive interaction (see Ref. 2). We show this phenomenon in the simplest case, $d=1$.

2. FORMULATION OF THE RESULTS

Consider the space

$$\Omega_n = \{\omega = (\omega(0),..., \omega(n)) \colon \omega(0)=0,\ \omega(i+1)-\omega(i) = \pm 1\}$$

of paths in $\mathbb{Z}^1$ starting at 0. On this space we introduce the following probability measure:

$$\mu_n(\{\omega\}) = Z_n^{-1} \exp[-\lambda k(\omega)]$$

where

$$k(\omega) = \#\{i \in \{1,..., n\} \colon \omega(i) = \omega(j) \text{ for some } j<i\}$$

If $\lambda=0$, then we have the standard random walk and Donsker's theorem asserts that $n^{-1/2}\omega([nt])$ converges to the Wiener process as $n\to\infty$ (here $[x]$ denotes the integer part of x). For $\lambda \neq 0$ we observe two different behaviors: if $\lambda>0$, then the generic path has a tendency to walk on one side of the origin (see Ref. 9); if $\lambda<0$, then the trajectory lies near the starting point. The results are expounded in the following theorems.

Theorem 1. If $\lambda < 0$, then the random variables $n^{-1/3}\omega(n)$ converge in a weak sense as $n \to \infty$ to the random variable with a density

$$\rho(x) = \frac{\pi}{8s}\left[\pi\left(1 - \frac{x}{s}\right)\cos\frac{\pi x}{s} + \sin\frac{\pi\,|x|}{s}\right]$$

for $|x| \leqslant s$, and $\rho(x) = 0$ otherwise, where $s = s(\lambda) = (\pi^2/|\lambda|)^{1/3}$.

The next theorem repeats in fact the result of Ref. 9. However, I know the formulation of the result of Ref. 9 only from a remark in Ref. 11, and am unaware of explicit formulas for asymptotic velocities and variances as a function of λ. For these reasons I formulate the result for $\lambda > 0$ and give the sketch of its proof in Section 5.

Theorem 2. (a) If $\lambda \geqslant 0$, then the random functions $t \to n^{-1}\omega([nt])$, $t \in [0, 1]$, converge in a weak sense as $n \to \infty$ to the random process ζ_t with the distribution $P(\{\zeta_t \equiv \pm rt\}) = \frac{1}{2}$, where $r = r(\lambda) = (e^{2\lambda} - 1)/(e^{2\lambda} + 1)$.

(b) Let $\lambda > 0$ and $\mu_{n+} = \mu_n(\bullet \mid \omega(n) > 0)$. Then the random functions $t \to \{(1 - r^2)n\}^{-1/2}[\omega([nt]) - rnt]$ converge in a weak sense with respect to the conditional measure μ_{n+} to the standard Wiener process.

Remarks. 1. One can use another form of the Gibbs factor

$$\tilde{\mu}_n(\omega) = \tilde{Z}_n^{-1}\exp\{-\lambda\tilde{k}(\omega)\}$$

where

$$\tilde{k}(\omega) = \#\{(i, j): i < j,\ \omega(i) = \omega(j)\}$$

This kind of interaction has been used by other authors. This is the so-called Edwards model. I suppose that the energy $k(\omega)$ is simpler in calculations. Moreover, the measure $\tilde{\mu}_n$ for fixed, negative λ is not stable. One must choose $\lambda \sim n^{-1}$ to guarantee stability. The measure μ_n is automatically stable because the energy of interaction $k(\omega)$ is less than n.

2. Obviously, the multidimensional case is the most interesting. It is expected and physically justified that for $\lambda > 0$

$$\begin{aligned} &\omega(n) \sim n^{3/4} && \text{for } d = 2 \\ &\omega(n) \sim n^{3/5} && \text{for } d = 3 \\ &\omega(n) \sim n^{1/2} && \text{for } d > 4 \end{aligned} \tag{1}$$

(see Refs. 2 and 7). Renormalization-group techniques should be useful to solve the problem.

3. Introducing spins $\sigma_i = \omega(i+1) - \omega(i)$, one can treat the model under consideration as a model of one-dimensional statistical mechanics. The energy of interaction in the volume $\Lambda = [0,..., n] \subset \mathbb{Z}^1$ is

$$H(\sigma_\Lambda) = \lambda k(\omega) = \lambda \sum_i \left(1 - \prod_{j<i} \operatorname{sign} |\sigma_j + \cdots + \sigma_{i-1}|\right)$$

This model undergoes a phase transition at $\lambda = 0$. The reason for this is that the radius of interaction is infinite (see also Ref. 13).

3. PRELIMINARY LEMMAS

In this section we prove some combinatorial formulas upon which the proofs of Theorems 1 and 2 are based.

Lemma 1. $k(\omega) = n - \{\sup_i \omega(i) - \inf_i \omega(i)\} - 1.$

Proof. Every point reached by a trajectory ω is first reached only once. Therefore, the number of moments when the self-intersection does not occur is equal to the number of points visited by ω, i.e., $\sup \omega(i) - \inf \omega(i) + 1$. This is equivalent to the assertion of Lemma 1.

The next result is very well known.

Lemma 2. The number of n-paths ω with $\sup \omega(i) < a$, $\inf \omega(i) > b$, and $\omega(n) = c$ is equal to

$$H_n(a, b, c) = \sum_{k=-\infty}^{\infty} \left(C_n^{\{n+2k(a-b)+c\}/2} - C_n^{\{n+2k(a-b)+2a-c\}/2}\right) \tag{2}$$

where $C_n^k = n!/\{k!\,(n-k)!\}$ is the binomial coefficient for $0 \leqslant k \leqslant n$, n, k integer, and $C_n^k = 0$ otherwise.

Proof. See Ref. 12, III, §10, Problem 3.

Lemma 3. The number of paths ω satisfying $\sup \omega = a$, $\inf \omega = b$, and $\omega(n) = c$ is

$$G_n(a, b, c) = -\partial_a \partial_b H_n(a, b-1, c) \tag{3}$$

where $\partial_x f(x) = f(x+1) - f(x)$.

Proof. This lemma is obvious.

4. PROOF OF THEOREM 1

Our basic tool is the Poisson summation formula,[15]

$$\sum_{k=-\infty}^{\infty} f(\alpha k+x)=\frac{(2\pi)^{1/2}}{\alpha}\sum_{k=-\infty}^{\infty}\tilde{f}\left(\frac{2\pi k}{\alpha}\right)e^{2\pi ikx/\alpha} \tag{4}$$

where $\tilde{f}$ is the Fourier transform of the function f. We apply the formula (4) to the function $f(z)=C_n^{[n/2+z]}$ with $\alpha=a-b$, $x=c/2$, or $x=a-c/2$ and n odd. By straightforward calculations we have

$$\tilde{f}(\xi)=\left(2\cos\frac{\xi}{2}\right)^n\frac{\sin(\xi/2)}{\xi/2}$$

Therefore

$$\sum_k C_n^{\{n+2k(a-b)+c\}/2}=\left(\frac{2}{\pi}\right)^{1/2}\sum_k\left(2\cos\frac{\pi k}{a-b}\right)^n\frac{\sin[\pi k/(a-b)]}{k}e^{\pi ick/(a-b)} \tag{5}$$

We return to the investigation of the measure μ_n for $\lambda<0$. Assume that $c\geqslant 0$. Using Lemmas 1–3, one can easily see that

$$\begin{aligned}\mu_n(\omega(n)=c)&=\text{const}\times\sum_{a\geqslant c,b\leqslant 0}G_n(a,b,c)\,e^{\lambda(a-b)}\\&=\text{const}\times\sum_{a\geqslant c,b\leqslant 0}\{-\partial_a\partial_b H_n(a,b-1,c)\}\,e^{\lambda(a-b)}\\&=\text{const}\times\sum_{a>c,b<0}e^{\lambda(a-b)}H_n(a,b,c)\\&=\text{const}\times\sum_{d=c+2}^{\infty}e^{\lambda d}\sum_{a=c+1}^{d-1}H_n(a,a-d,c)\end{aligned} \tag{6}$$

By (2) and (5) we have

$$\begin{aligned}\sum_{a=c+1}^{d-1}&H_n(a,a-d,c)\\&=\left(\frac{2}{\pi}\right)^{1/2}\sum_{k=1}^{\infty}\left(2\cos\frac{\pi k}{d}\right)^n\frac{\sin(\pi k/d)}{k}\\&\quad\times\left\{(d-c-2)\cos\left(\frac{\pi kc}{d}\right)+\frac{\sin\{\pi k(c+2)/d\}}{\sin(\pi k/d)}\right\}\\&=\left(\frac{2}{\pi}\right)^{1/2}\sum_{k=1}^{\infty}\left(2\cos\frac{\pi k}{d}\right)^n\frac{1}{k}\left\{\frac{d-c-2}{2}\sin\frac{\pi k(c+1)}{d}\right.\\&\quad\left.-\frac{d-c-2}{2}\sin\frac{\pi k(c-1)}{d}+\sin\frac{\pi k(c+2)}{d}\right\}\end{aligned} \tag{7}$$

We denote $d=[\eta n^{1/3}]$ and $c=[\nu n^{1/3}]$. As $n\to\infty$ only the term with $k=1$ in the sum in the right side of (7) predominates. This follows from the estimates

$$\begin{aligned}\Bigg|\sum_{l=1}^{\infty}\left(\cos\frac{\pi}{d}\right)^n(-1)^l\Bigg\{&\frac{\sin\{\pi(ld-1)(c+r)/d\}}{ld-1}\\&+\frac{\sin\{\pi(ld+1)(c+r)/d\}}{ld+1}\Bigg\}\Bigg|\\&\leqslant\left(\cos\frac{\pi}{d}\right)^n\sum_{l=1}^{\infty}\frac{2}{l^2d^2}\leqslant\frac{K}{d^2}\left|\cos\frac{\pi}{d}\right|^n\end{aligned}$$

and

$$\begin{aligned}&\Bigg|\left(\cos\frac{\pi s}{d}\right)^n\Bigg\{\frac{\sin\{\pi s(c+r)/d\}}{s}\\&\qquad-\sum_{l=1}^{\infty}(-1)^l\left[\frac{\sin\{\pi(ld-s)(c+r)/d\}}{ld-s}+\frac{\sin\{\pi(ld+s)(c+r)/d\}}{ld+s}\right]\Bigg\}\Bigg|\\&\leqslant\left|\cos\frac{\pi s}{d}\right|^n\left\{2+\sum_{l=1}^{\infty}\frac{2s}{l^2d^2-s^2}\right\}\\&\leqslant K\left|\cos\frac{\pi}{d}\right|^{(1+\varepsilon)n},\qquad s=2,3,\ldots,\frac{d}{2}\end{aligned}$$

where $r=\pm 1, 2$ and $\varepsilon>0$, $K>0$ do not depend on n, c, d, or s. Therefore, from (6) and (7) we obtain

$$\begin{aligned}&\mu_n(\omega(n)=[\nu n^{1/3}])\\&\quad=\text{const}\times 2^n n^{1/3}\int_\nu^\infty d\eta\,[\exp(\lambda\eta-\tfrac{1}{2}\pi^2/\eta^2)]^{n^{1/3}}\\&\qquad\times[\pi(1-\nu/\eta)\cos(\pi\nu/\eta)+\sin(\pi\nu/\eta)][1+o(1)]\end{aligned}\tag{8}$$

as $n\to\infty$. The function $\exp(\lambda\eta-\frac{1}{2}\pi^2/\eta^2)$ takes its maximal value at the point $\eta=s(\lambda)=(\pi^2/|\lambda|)^{1/3}$. Hence, the integral in (8) is proportional to $\pi(1-\nu/s)\cos(\pi\nu/s)+\sin(\pi\nu/s)$. From this and from (8), Theorem 1 follows.

5. PROOF OF THEOREM 2

First we concentrate on the proof of the formulas

$$\lim\omega(n)/n=\pm r,\qquad \lim[\omega(n)-rn]/[(1-r^2)n]^{1/2}=N(0,1)$$

By Lemma 1 one must consider the joint distribution of the endpoint and the extreme points of a random path (similarly as in Ref. 9). Let $a=[nx]$, $b=[ny]$, $c=[nz]$, $x>y\leqslant 0\leqslant x\geqslant z\geqslant y$, $x-y-|z|/2<1/2$. By (3),

$$G_n(a, b, c)=-\frac{1}{n^2}\frac{\partial^2 H_n(a, b, c)}{\partial_x \partial_y}\left[1+O\left(\frac{1}{n}\right)\right]$$

We shall find the asymptotic formula for $H_n(a, b, c)$. The number of summands in (2) is finite. Hence it is enough to find the asymptotic formula for the greatest one. The greatest nonvanishing term after differentiation is $C_n^{\{n+2(a-b)-|c|\}/2}$.

Next we use the formula

$$\begin{aligned}(C_n^k)^{-1}&=(n+1)\frac{\Gamma(k+1)\,\Gamma(n-k+1)}{\Gamma(n+2)}\\ &=(n+1)\,B(k, n-k)\\ &=(n+1)\int_0^1 \tau^k(1-\tau)^{n-k}\,d\tau\end{aligned}$$

where B is the beta function.[14] To compute the integral (9) we use the Laplace method. Finally we get

$$\begin{aligned}&\mu_n(\inf\omega=b, \sup\omega=a, \omega(n)=c)\\ &\quad=\exp[-n\gamma(x, y, z)+\text{const}\times\ln n+O(1)]\end{aligned}$$

where

$$\begin{aligned}\gamma(x, y, z)=&\left(\frac{1}{2}+x-y-\frac{|z|}{2}\right)\ln\left(\frac{1}{2}+x-y-\frac{|z|}{2}\right)\\ &+\left(\frac{1}{2}+y-x+\frac{|z|}{2}\right)\ln\left(\frac{1}{2}+y-x+\frac{|z|}{2}\right)-\lambda(x-y)\end{aligned}$$

Straightforward calculations show that for $\lambda>0$ the function γ (considered in the domain $D=\{x<y\leqslant 0\leqslant x\geqslant z\geqslant y,\ x-y-|z|/2<1/2\}$, takes its minimal value at the points P_+: $y=0$, $x=z=r$ and P_-: $y=z=-r$, $x=0$ in the boundaries $S_+=\{y=0, x=z\}$ and $S_-=\{x=0, y=z\}$ and

$$(\gamma\,|_{S_\pm})''(P_\pm)=1/(1-r^2)$$

From this the asymptotic formulas for the end of a random path easily follow.

In order to prove the asymptotic behavior of the whole random path one divides the interval $[0, 1]$ by the points $0 = t_0 < t_1 < \cdots < t_s = 1$ and computes the joint distribution of the variables

$$\omega([nt_i]), \qquad \sup_{[nt_j] \leqslant i \leqslant [nt_{j+1}]} \omega(i), \qquad \inf_{[nt_j] \leqslant i \leqslant [nt_{j+1}]} \omega(i)$$

The further analysis is only a slight complication of that presented above for the endpoint of the path ω. We thus complete the proof of Theorem 2.

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